Equation of state for a classical gas of BPS black holes

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Abstract

A point-particle treatment for the statistical mechanics of Bogomol'nyi-Prasad-Sommerfield (BPS) black holes in Einstein-Maxwell-dilaton theory is developed. Because of the absence of the static potential, the canonical partition function for $N$ BPS black holes can be expressed by the volume of the moduli space for them. We estimate the equation of state for a classical gas of BPS black holes by the Padé approximation and find that the result agrees with the one obtained by the mean-field approximation.

1 Introduction

The thermodynamics of a self-gravitating system has some importance in the context of cosmology. The long-range nature of gravity gives rise to the breakdown of the conventional description statistical properties, such as the additivity of the free energy or other "extensive" thermodynamical quantities. While a statistical treatment of nonequilibrium systems has recently been developed, the existence of the long-range force seems essential for explaining the formation of the peculiar local structure in the system.

Recently, de Vega and Sánchez studied the statistical mechanics of a self-gravitating particle gas [2]. They showed that the "thermodynamic limit" must be taken as $N \to \infty$ with $N/V^{1/3}$ fixed in three dimensions, where $N$ is the number of particles and $V$ is the volume of the system. This treatment is required by the long-range nature of gravity, because there is no ad hoc cutoff scale.\textsuperscript{5}

On the other hand, two of the present authors have studied the statistical mechanics of well-separated charged particles in Einstein-Maxwell-scalar theory [3]. The system was found to be unstable in general, although there exists a critical case in which the static forces cancel each other.

In general relativity, the critical case was investigated and exact static solutions which describe multiblack hole systems have been obtained [4, 5]. In such a case, the attractive forces (the gravitational and scalar-mediated forces) and the repulsive forces (the electro-static force) between black holes are exactly cancelled in the static limit. In this case, the energy of the system was calculated for small velocities but for small distances between the black holes [6, 7]

In this paper, we investigate the multi-black hole system by adopting the technique of de Vega and Sánchez [2]. Note that a mass-charge relation is satisfied for each individual black hole. Such extreme black holes, or so-called BPS black holes,\textsuperscript{6} often appear in string theories.

\textsuperscript{1}This work is based on [1].
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\textsuperscript{5}Actually, the horizon length is a candidate for the cutoff scale in the universe.
\textsuperscript{6}In [8], it is shown that they saturate a gravitational analogue of the Bogomol'nyi bound on their mass and charges.
2 “BPS black holes” in slow motion

We consider ‘BPS black holes’ in \((d + 1)\) dimensional Einstein-Maxwell-dilaton theory, which is governed by the action

\[
S = \int d^{d+1}x \frac{\sqrt{-g}}{16\pi G} \left[ R - \frac{4}{d-1} \nabla_\mu \phi \nabla^\mu \phi - e^{-\frac{4\alpha}{(d-1)}} \phi \right] F_{\mu\nu} F^{\mu\nu},
\]

(1)

Here \(R\) is the scalar curvature and \(F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \) \((\mu, \nu = 0, 1, \ldots, d)\), while \(\phi\) is a dilaton field and \(a\) is a coupling constant. \(G\) denotes the \((d + 1)\) dimensional Newton’s constant. We set \(c = 1\).

This theory admits static multi-centered solutions whose metric and field configurations are given by [5]

\[
ds^2 = -V^{-2(d-2)/(d-2+a^2)} dt^2 + V^{2/(d-2+a^2)} dx^2,
\]

(2)

\[e^{-\frac{4\alpha}{(d-1)} \phi} = \frac{V^{2a^2/(d-2+a^2)} \left(1 - \frac{1}{V}\right)}{2(d-2+a^2)},\]

(3)

\[A_0(x) = \sqrt{\frac{d-1}{2(d-2+a^2)}} \left(1 - \frac{1}{V}\right),\]

(4)

\[A_i(x) = 0 \quad (i = 1, \ldots, d),\]

(5)

in terms of a harmonic function

\[V(x) = 1 + \frac{2(d - 2 + a^2)}{(d - 1)(d - 2)} \frac{4\pi}{A_{d-1}} \sum_a \frac{m_a}{|r_a|^{d-2}},\]

(6)

where \(r_a = x - x_a\) is the position vector of ‘point particle’ \(a\) and the labels of ‘particles’ \(a, b, \ldots\) run over \(1, \ldots, N\). \(A_{d-1} \equiv 2\pi^{d/2}/\Gamma(d/2)\) is the volume of a unit \(d - 1\) sphere.

The point sources corresponding to the ‘particles’ can be described by the action

\[I = -\sum_a \int ds_a \left[ m_a e^{-\frac{4\alpha}{(d-1)} \phi} - e_a A_\mu \frac{dx^\mu}{ds_a}\right],\]

(7)

with the relation

\[\frac{e_a}{m_a} = \sqrt{\frac{2(d - 2 + a^2)}{d - 1}}.\]

(8)

This relation is just the extremity condition for spherically symmetric black holes in the Einstein-Maxwell-dilaton theory [9]. Thus the solution can be viewed as the one for an \('N BPS black hole'\) system, where the electric forces is cancelled by the forces mediated by the gravitation and the dilaton.

3 The canonical partition function

The canonical partition function for \(N\) identical particles at temperature \(T\) in \(d\) spatial dimensions is

\[Z_N = \frac{1}{\sqrt{2\pi}} \left(\frac{1}{h}\right)^{d/2} \int \frac{[dp]}{[dq]} \exp(-\beta H),\]

(9)

where \([dp] = \Pi_{a=1}^N d^dp_a\), \([dq] = \Pi_{a=1}^N d^dx_a\), and \(\beta = 1/T\). \(h\) is Planck’s constant while Boltzmann’s constant is set to unity.

Because there are no static forces, the Hamiltonian \(H\) for the system of \(N\) “BPS black holes” with a common mass \(m\) can be written as

\[H = \frac{1}{2} \sum \sum \left(\delta_{AB} v^A G_{AB} v^B\right),\]

(10)

where

\[G_{(ak)(bl)} = m \left[\delta_{ab} \delta_{kl} + \frac{2}{m} \left(\delta_{kl} \delta_{ij} + \delta_{jk} \delta_{li} - \delta_{ij} \delta_{lj}\right) \partial_{ai} \partial_{bj} L\right] \equiv m g_{(ak)(bl)},\]

(11)
with
\[ L' = -\frac{1}{32\pi G} \int d^4x \left[ V^{2(d-1)/(d-2+a^2)}(x) - 1 - \frac{4}{d-2} \frac{4\pi}{A_{d-1}} Gm \sum_c \frac{1}{|r_c|^d} \right], \tag{12} \]
and
\[ V(x) = 1 + \frac{2(d-2+a^2)}{(d-1)(d-2)} \frac{4\pi}{A_{d-1}} Gm \sum_a \frac{1}{|r_a|^d}. \tag{13} \]
The label \( \{A\} = \{a_1\} \) denotes the combination of the label of the particle and the spatial index. Then we obtain the expression for the partition function
\[ Z_N = \frac{1}{N!} \left( \frac{2\pi m}{\beta h^2} \right)^{Nd/2} \int [dq] \sqrt{\det g} = \frac{1}{N!} \left( \frac{2\pi m}{\beta h^2} \right)^{Nd/2} Q, \tag{14} \]
which is proportional to the volume of the moduli space \( Q \).

4 The thermodynamical limit and the equation of state

The pressure \( P \) is derived from the free energy \( F \) as \( P = -\frac{\partial F}{\partial V} \), where \( V = \int d^4x = A_{d-1} l^d/d \), \( l \) is a radius of a spherical box, thus we find
\[ \frac{PV}{NT} = 1 + \frac{V}{N} \frac{\partial}{\partial V} \ln \frac{Q}{V^N} = 1 + \frac{l}{Nd \partial l} \ln \frac{Q}{V^N}. \tag{15} \]
Substituting the above result and taking the thermodynamical limit, or in the large-\( N \) limit,
\[ W = \lim_{N \to \infty} \frac{1}{N} \ln \frac{Q}{V^N} = W(y), \tag{16} \]
where the value \( Q/l^{d-2} \) is fixed, we obtain the equation of state
\[ \frac{PV}{NT} = f(y), \tag{17} \]
where
\[ y = \frac{1}{d-2} \frac{4\pi}{A_{d-1}} \frac{GmN}{l^{d-2}} \] \( \tag{18} \)
is a dimensionless parameter. We estimate \( f(y) \) by the Padé approximation and by the mean-field approximation.

5 The Padé approximation

The Padé approximation interpolates between the small and large \( y \) regions.

For \( y \ll 1 \), weakly self-interacting gas, the equation of state reads
\[ f(y) = 1 - b_1 y + b_2 y^2 + O(y^3) \tag{19} \]
where
\[ b_1 = \frac{d-2}{2} \left[ \frac{2(d-a^2)}{d-1} \right] \frac{2d}{d+2}, \tag{20} \]
\[ b_2 = d(d-2) \left[ \frac{2(d-a^2)}{d-1} \right]^2 \left[ \left( \frac{2d}{d+2} \right)^2 - \left( 1 - \frac{d-2+a^2}{2d(d-a^2)} \right) \frac{d(5d+2)}{(d+2)(d+4)} \right]. \tag{21} \]
In the strong coupling limit \( y \gg 1 \), we find
\[ \frac{Q}{V^N} \approx y^{(Nd/2)(d-a^2)/(d-2+a^2)} \times \text{const.} \tag{22} \]
Then the Padé approximation shows that
\[
\frac{Q}{V^N} \approx \left[ q_{\text{in}}(y) \right]^N = (1 + \alpha_1 y)^{(Nd/2)(d-a^2)/(d-2+a^2)},
\]
(23)
with
\[
\alpha_1 = \frac{2(d-2+a^2)}{d-1} \frac{2d}{d+2}.
\]
(24)
In this approximation, the equation of state reads
\[
\frac{PV}{N^T} = f(y) = 1 - \frac{(d-2)(d-a^2)}{d-1} \frac{2d}{d+2} \frac{\alpha_1 y}{1 + \alpha_1 y}.
\]
(25)
At the critical \( y = y_c \) [\( y_c \) satisfies \( f(y_c) = 0 \)] the gas cannot remain in the normal gaseous phase.  

6 The mean-field approximation

In [7], an effective field theory of "BPS black holes" was constructed. Here we use the effective field theory to obtain the mean-field approximation.

In the mean-field approximation the number density with a chemical potential \( \mu \) is given by
\[
n = \frac{1}{h^d} \int d^d p \exp \left[ -\beta \left( \frac{p^2}{2mV(d-a^2)/(d-2+a^2)} - \mu \right) \right] = n_0 V^{d(d-a^2)/(d-2+a^2)},
\]
(26)
with a constant \( n_0 \), where the gauge interaction term is discarded.

The mean-field potential \( V \) satisfies the following equation [7]:
\[
\partial^2 V + 8\pi \frac{d-2+a^2}{d-1} G m n_0 V^{d(d-a^2)/(d-2+a^2)} = 0,
\]
(27)
with the boundary conditions.

The canonical partition function can be expressed as
\[
Z_N = \frac{V^N}{N!} \left[ \frac{1}{V} \int d^d x \left( \frac{2\pi m}{\beta h^2} V^{(d-a^2)/(d-2+a^2)} \right)^{d/2} \right]^N = \frac{V^N}{N!} \left( \frac{2\pi m}{\beta h^2} \right)^{Nd/2} \left[ q_{mf}(y) \right]^N.
\]
(28)
Then the equation of state reads
\[
\frac{PV}{N^T} = f(y) = 1 - \frac{d-2}{d} \frac{\partial}{\partial y} \ln q_{mf}(y).
\]
(29)
We evaluate \( f(y) \) by a numerical method, since the solution cannot be given analytically in general.

7 Result

In Fig. 1, \( f(y) \) obtained by the approximations considered above is shown. All the mean-field and the Padé approximations indicate consistent behavior of the function. The difference appears to be small when \( d-a^2 \) is small.

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\*In the present approximation scheme, the value of \( y = y_c \) is \( y_c = \frac{d-a^2}{d-2} \frac{d-2}{d-a^2} \), for \( a^2 < a_c^2 \equiv (d-2)^2/d \).
Figure 1: $f(y)$ is plotted against $y$ for $d = 3$ and $d = 4$. The solid line obtained by the mean-field approximation, the dashed line by the lowest-order Padé approximation, the dot-dashed line by the next-to-leading Padé approximation, and the dotted line is the series expansion up to second order.

8 Summary

We have studied a many-body system of “BPS black holes” in $d$ dimensions. The canonical partition function of the system is proportional to the volume of the moduli space of $N$ “BPS black holes”. Therefore the difference from an ideal gas arises through the change of the effective volume of the system: the internal energy and heat capacity at a fixed volume are the same as those of an ideal gas [1].

We estimated the equation of state of the gas by the Padé approximation and the mean-field method. In these approximations, we found that the pressure simply decreases as the value of $y$ increases if the temperature and the density are fixed. This feature is common in the cases for different spatial dimensions and dilaton couplings.

Of course the analytical attempt to describe thermodynamic properties of the “BPS black hole” gas may have been inconclusive and numerical simulations of the self-interacting system may be needed. The present work provides a guideline for such numerical calculations to survey the critical points of the “BPS black hole” gas.

References