Low-Energy Interaction of a Cosmic String and an Extreme Dilatonic Black Hole \footnote{Talk given at the fifth workshop on GRG, Nagoya University, 1/22-25/1996.} \footnote{This talk is based on the work gr-qc/9512001.}

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We construct the solitonic solution in Einstein-Maxwell-dilaton theory on the conical spacetime and investigate the motion of the solitonic object at a low velocity. In the following, using the method of images, we obtain a solitonic solution on a conical space and the low-energy interaction.\footnote{The multi-soliton system in Einstein-Maxwell-dilaton theory on the spacetime with torus compactification has been studied by obtaining exact solutions and low energy interactions in a similar method \cite{1}.}

\section{WHAT IS A "COSMIC STRING"?}

The spacetime outside of a straight, thin cosmic string is described by the metric \cite{2}

\begin{equation}
    ds^2 = -dt^2 + dr^2 + \frac{r^2}{\nu^2} d\theta^2 + dz^2 ,
\end{equation}

with $\nu = (1 - 4G\mu)^{-1}$, where $\mu$ is the mass density per unit length.

If we use a new coordinate
\[ \theta = \frac{\theta}{\nu} \quad 0 \leq \theta < \frac{2\pi}{\nu} \] , \hspace{1cm} (2)

the metric (1) reduces to the one of the flat spacetime

\[ ds^2 = -dt^2 + dr^2 + r^2 d\theta^2 + dz^2 . \] \hspace{1cm} (3)

Then the deficit angle \( \Delta \) is defined as [2]

\[ \Delta \equiv 2\pi \left(1 - \frac{1}{\nu}\right) = 8\pi G \mu . \] \hspace{1cm} (4)

II. WHAT IS AN "EXTREME DILATONIC BLACK HOLE"?

We consider a model described by the following action:

\[ S = \int d^4 x \frac{\sqrt{-g}}{16\pi G} \left[ R - 2(\nabla \phi)^2 - e^{-2\phi} F^2 \right] + \text{(surface terms)} . \] \hspace{1cm} (5)

where \( R \) is the scalar curvature, \( \phi \) a dilaton, and \( F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \) is the abelian gauge field strength.

We assume that the electric solution in the Einstein-Maxwell-dilaton theory takes the form [3,4]

\[ ds^2 = -\frac{1}{V} dt^2 + V dx^2 , \quad A_\mu dx^\mu = \frac{1}{\sqrt{2}} \left(1 - \frac{1}{V}\right) dt , \quad e^{-2\phi} = V . \] \hspace{1cm} (6)

Then \( V \) satisfies

\[ \nabla^2 V = 0 , \] \hspace{1cm} (7)

up to a number of delta functions in the right hand side of Eq. (7).

For example, one can find a solution:

\[ V = 1 + \sum_\alpha \frac{2Gm_\alpha}{|x - x_\alpha|} . \] \hspace{1cm} (8)

By studying the relation among the electric and dilatonic charge and mass of each solitonic object, we find that the solution describes the configuration that the \( \alpha \)-th nonrotating, charged dilatonic black hole in the extreme limit with mass \( m_\alpha \) located at \( x = x_\alpha \) [3,4].
Now let us construct the black hole solution on a conical space. [5] 2

When the string parameter $\nu$ is equal to an integer $p = 1, 2, 3, \ldots$, we can derive the solution for $V$, which corresponds to the static configuration of one extreme dilatonic black hole and a cosmic string at the origin, by the method of images [6]. The function $\Gamma_p(x, x')$ satisfying Eq. (7) and having periodicity $2\pi/p$ in $\vartheta$ and $\vartheta'$ can be expressed by the summation of $p$ images of $\Gamma(x, x')$, which is a solution of Eq. (7) in the Minkowski space:

$$\Gamma_p(x, x') = \sum_{n=0}^{p-1} \Gamma(x, \Lambda^n x') = \sum_{n=0}^{p-1} \frac{1}{|x - \Lambda^n x'|},$$

(9)

where $\Lambda$ means the rotation by $2\pi/p$ around the origin. Using the integral representation, we then obtain the solution for $V$ as follows:

$$\Gamma_p(x, x') = \sum_{n=0}^{p-1} \frac{1}{r^2 + r'^2 - 2rr' \cos(\vartheta - \vartheta' - 2\pi n/p) + (z - z')^2}
= \frac{1}{\pi \sqrt{2rr'}} \int_{u_0}^{\infty} \frac{du}{\sqrt{\cosh u - \cosh u_0 \cosh pu - \cosh pu - \cos p(\vartheta - \vartheta')}} \frac{p \sinh pu}{},$$

(10)

where

$$\cosh u_0 = \frac{r^2 + r'^2 + (z - z')^2}{2rr'}.\quad(11)$$

We have the solution of $V$ which corresponds to asymptotically flat space:

$$V = 1 + 2GM\Gamma_p(x, x'),$$

(12)

where $M$ is a constant. Note that $V$ is a solution of Eq. (7) even if $p$ is replaced by an arbitrary real number $\nu$. Consequently, we have the solution of $V$ corresponding to an extreme dilatonic black hole located at $x' = (r', \vartheta', z')$ in the conical space:

$$V = V_{CS} = 1 + \frac{2GM}{\pi \sqrt{2rr'}} \int_{u_0}^{\infty} \frac{du}{\sqrt{\cosh u - \cosh u_0 \cosh \nu u - \cos(\vartheta - \vartheta')}} \frac{\nu \sinh \nu u}{},$$

(13)

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2The cosmic string-extreme black hole solution in a general Einstein-Maxwell-dilaton model with an arbitrary dilaton coupling [4] can be constructed in the same manner.
This result can be extracted by using the mode expansion in terms of the Legendre functions. \( V_{CS} \) can be equivalently expressed as the sum of the Legendre functions:

\[
V_{CS} = 1 + \frac{2GM\nu}{\pi \sqrt{r'r'}} \sum_{n=-\infty}^{\infty} Q_{n[r]}(\cosh u_0) e^{in(\theta - \theta')}
\]  

(14)

In the limit of \( \nu \to 1 \), \( V_{CS} \) is reduced to \( V \) in the flat space [3,4]. On the other hand, in the limit of \( r' \to 0 \), \( V_{CS} \) is reduced to

\[
V = 1 + \frac{2GM\nu}{\sqrt{r^2 + (z - z')^2}}
\]  

(15)

which gives the metric representing a cosmic string trapped by a black hole. Thus the constant \( M \) is identified to the black hole mass [7].

III. WHAT IS "LOW-ENERGY INTERACTION"?

Next we will present the low-energy interaction in the cosmic string-extreme black hole system.

The interaction energy of the maximally charged dilatonic black holes in \( R^4 \) at low velocities has been calculated without a long-distance approximation by making use of the exact, static solution (6) with (8) [8,9]. Since there are only two-body velocity-dependent forces in the multi-black hole system in this case, the general expression for the interaction energy of \( O(\nu^2) \) of an arbitrary number of extreme dilatonic black holes can be obtained as: [8–10]

\[
H = \frac{1}{2} \mathcal{M} V^2 + \sum_{\alpha\beta} \frac{m_\alpha m_\beta |v_\alpha - v_\beta|^2}{4\mathcal{M}} \left( 1 + \frac{2GM}{|x_\alpha - x_\beta|} \right),
\]  

(16)

where \( v_\alpha \) is the velocity of the extreme dilatonic black hole with mass \( m_\alpha \). \( \mathcal{M} \) is the total mass, \( \mathcal{M} = \sum_\alpha m_\alpha \). \( V \) is the velocity of the center of mass; \( V \equiv \frac{1}{\mathcal{M}} \sum_\alpha m_\alpha v_\alpha / \mathcal{M} \).

We use the method of images to obtain the interaction energy of extreme dilatonic black holes. We must note that the images of a certain black hole are located at \( \Lambda^\infty x \) and have the same mass \( M \) and the velocity \( \Lambda^\infty v \). We also note that the sum of the interaction energy
must be divided by the number of the images. The number \( p \) is extended to a real number \( \nu \) after completing the calculation.

Finally we get the following interaction energy up to \( O(\nu^2) \) of the extreme dilatonic black hole with the mass \( M \) and the velocity \( \nu \) on the conical space:

\[
H_{CS} = \frac{1}{2} M v_{||}^2 + \frac{1}{2} M \left[ 1 + \frac{2GM}{\rho} \tan \frac{\Delta}{4} \right] v_{\perp}^2 ,
\]  

(17)

where \( \rho \) is the distance between the cosmic string and the extreme black hole. \( v_{||} \) is the \( z \) component of the velocity of the extreme dilatonic black hole, while \( v_{\perp} \) is the component of the velocity perpendicular to the cosmic string. It is quite reasonable that the interaction energy diverges in the limit of \( \Delta \to 2\pi \).

IV. SCATTERING OF AN EXTREME DILATONIC BLACK HOLE BY A COSMIC STRING

We consider the scattering of a maximally charged dilatonic black hole by a cosmic string by using the metric of moduli space [11].

We assume that the black holes move in a plane which is perpendicular to the string. Therefore the moduli space of this configuration is reduced to be a two-dimensional space parameterized by the distance \( \rho \), the azimuthal angle \( \varphi \). For this two-body system, the metric on moduli space which spanned by the coordinate \( \xi \) can be read from Eq. (17) as

\[
ds^2_{MS} = \gamma_{ij} d\xi^i d\xi^j = \gamma(\rho) \left( d\rho^2 + \frac{\rho^2}{\nu^2} d\varphi^2 \right) ,
\]  

(18)

with

\[
\gamma(\rho) = 1 + \frac{2GM}{\rho} \tan \frac{\Delta}{4} = 1 + \frac{2\rho_0}{\rho} .
\]  

(19)

\footnote{For a case of a general dilaton coupling, since the interaction contains many-body, velocity-dependent forces [8–10], the expression will be more complicated.}
Here $\varphi$ stands for the azimuthal angle which has a range $0 \leq \varphi < 2\pi$.

The path of the moving extreme black hole is determined by the geodesic equation on the moduli space [11], because the geodesic on the metric (18, 19) realizes the path of minimal energy. We find that the scattering trajectory satisfies the following differential equation:

$$
\left( \frac{du}{d\varphi} \right)^2 + \nu^2 u^2 = \frac{1}{b^2} \left( 1 + 2\rho_0 u \right),
$$

where $u = 1/\rho$ and $b$ is the impact parameter.

The scattering angle $\Theta$ can be obtained by solving the equation (20) and expressed as:

$$
\frac{\Theta}{\nu} = \frac{\Delta}{2} + 2 \arctan \left( \frac{\rho_0 \nu}{b} \right).
$$

The angle of deflection is written as the sum of the contributions from the deficit angle [2] and the Rutherford scattering [9].

The quantum mechanical approach to the study of the scattering process is also possible by using the moduli space metric [12,9]. The Schrödinger equation on the moduli space, which is spanned by the coordinates $(\rho, \varphi, Z)$, can be read as [12,9]

$$
i\hbar \frac{\partial \Psi}{\partial t} = -\frac{\hbar^2}{2M} \left( \frac{1}{\sqrt{\det \gamma_{ij}}} \gamma^{ij} \partial_t + \frac{\partial^2}{\partial Z^2} \right) \Psi.
$$

When we assume the wave function $\Psi$ can be decomposed as

$$
\Psi = \psi(\rho, \varphi) e^{-i \frac{\hbar}{\nu} t} e^{i k Z},
$$

we get the following wave equation:

$$
\left[ \frac{1}{\rho} \frac{\partial}{\partial \rho} \rho \frac{\partial}{\partial \rho} + \frac{\nu^2}{\rho^2} \frac{\partial^2}{\partial \varphi^2} + q^2 \left( 1 + \frac{2\rho_0}{\rho} \right) \right] \psi = 0,
$$

where $q^2 \equiv 2ME/\hbar^2 - k^2$. The equation (24) is the same as Eq. (4.2) in the paper on the Coulomb problem on a cone by Gibbons, Ruiz and Vachaspati [13], if we replace $\rho \rightarrow r$, $\varphi \rightarrow \nu \phi$, $q \rightarrow k$ and $q^2 \rho_0 \rightarrow \mu K$. Thus we do not repeat the calculations on the scattering problem here. One can find that the differential cross-section diverges in the forward direction, but the divergence is shifted due to the presence of deficit angle of the spacetime [13]. This nature agrees with that of the classical scattering.


