

# Instability in self-gravitating systems of particles with nearly critical charges

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## 1 Introduction

The physics of self-gravitating particle system has been investigated not only by numerical simulations but also by analytic studies. We can consider the case that there are other long-range forces acting between particles; for instance, the electromagnetic forces, and/or the force mediated by a massless scalar field. In general, elementary particles or states which appear in recent models of unified theory have various interactions other than gravitation. Thus, the study of the system of the particles with many kinds of “charges” may reveal some aspects of the very early stage, or beginning of our universe.

Further, recent studies on string theories and so-called M theory have shown that the elementary excitations correspond to BPS or nearly BPS black holes. Among the BPS black holes, static forces are cancelled out. There is, however, velocity-dependent forces among them. When the many particle system of (nearly) BPS states is studied, it must be important to take the velocity-dependent interactions into account.

In this talk, we consider the extension of the method of gaussian integration used by de Vega et al. [1] to the case in the presence of velocity-dependent interactions.

## 2 Review of gaussian integration technique

One can write the Hamiltonian of  $N$ -particle system with the Newtonian potential as

$$H_N = \sum_{\ell=1}^N \frac{p_\ell^2}{2m} - Gm^2 \sum_{\ell < j} \frac{1}{|\vec{q}_\ell - \vec{q}_j|}. \quad (1)$$

Then the grandcanonical partition function is written as

$$\begin{aligned} Z_G &= \sum_{N=0}^{\infty} \frac{z^N}{N!} \iint \prod_{\ell=1}^N \frac{d^3 p_\ell d^3 q_\ell}{(2\pi)^3} e^{-\beta H_N} \\ &= \sum_{N=0}^{\infty} \frac{1}{N!} \left[ z \left( \frac{m}{2\pi\beta} \right)^{3/2} \right]^N \iint \prod_{\ell=1}^N d^3 q_\ell \exp \left( \beta Gm^2 \sum_{\ell < j} |\vec{q}_\ell - \vec{q}_j|^{-1} \right), \end{aligned} \quad (2)$$

where  $z$  is the activity. Here we introduce the density function

$$\rho(\vec{r}) = \sum_{j=1}^{\infty} \delta(\vec{r} - \vec{q}_j) \quad (3)$$

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to perform Stratonovich-Hubbard transformation, by using the following gaussian integration:

$$\begin{aligned} & \exp \left( \frac{1}{2} \beta G m^2 \int \frac{d^3 x d^3 y}{|\vec{x} - \vec{y}|} \rho(\vec{x}) \rho(\vec{y}) \right) \\ &= \iint [D\xi] e^{-\frac{1}{2} \int d^3 x (\nabla \xi)^2 - 2m \sqrt{\pi G \beta} \int d^3 x \xi(\vec{x}) \rho(\vec{x})}. \end{aligned} \quad (4)$$

Then the grandcanonical partition function becomes

$$\begin{aligned} Z_G &= \sum_{N=0}^{\infty} \frac{z^N}{N!} \left( \frac{m}{2\pi\beta} \right)^{3N/2} \int [D\xi] e^{-\frac{1}{2} \int d^3 x (\nabla \xi)^2} \iint \prod_{\ell=1}^N d^3 q_{\ell} e^{-2m \sqrt{\pi\beta G} \sum_{\ell=1}^N \xi(\vec{q}_{\ell})} \\ &= \int [D\xi] e^{-\frac{1}{2} \int d^3 x (\nabla \xi)^2} \sum_{N=0}^{\infty} \frac{z^N}{N!} \left( \frac{m}{2\pi\beta} \right)^{3N/2} \left[ \int d^3 q e^{-2m \sqrt{\pi\beta G} \xi(\vec{q})} \right]^N \\ &= \int [D\xi] \exp \left\{ - \int d^3 x \left[ \frac{1}{2} (\nabla \xi)^2 - z \left( \frac{m}{2\pi\beta} \right)^{3/2} e^{-2m \sqrt{\pi\beta G} \xi} \right] \right\}. \end{aligned} \quad (5)$$

From the partition function, for example, we can derive the density of the particles in this system:

$$\rho = z \frac{\partial}{\partial z} \ln Z_G = z \left( \frac{m}{2\pi\beta} \right)^{3/2} \left\langle e^{-2m \sqrt{\pi\beta G} \xi} \right\rangle, \quad (6)$$

where  $\langle \rangle$  means the field theoretic expectation value.

In this formulation, we can examine the properties of the physical quantities in the system by using the various field-theoretical method [1].

As an application, we can derive equations for the static equilibrium configuration. Such equations are obtained by the evaluation of the steepest descend, or the variation of the effective action. One can get the following equation from (5):

$$\nabla^2 \xi = z \left( \frac{m}{2\pi\beta} \right)^{3/2} 2m \sqrt{\pi\beta G} \exp \left( -2m \sqrt{\pi\beta G} \xi \right). \quad (7)$$

If we identify the gravitational potential  $\phi$  as

$$\phi = 2 \sqrt{\frac{\pi G}{\beta}} \xi, \quad (8)$$

then we obtain

$$\nabla^2 \phi = z \left( \frac{m}{2\pi\beta} \right)^{3/2} 4\pi G m \exp(-\beta m \phi) = 4\pi G m \rho. \quad (9)$$

This equation is the well-known one for the gravitating isothermal sphere [2].

### 3 Incorporation of velocity-dependent interaction

Now we turn to study of the velocity-dependent interaction. First, we consider gravity only. The potential up to  $O(v^2)$  is given by [3]

$$\begin{aligned}
 V = & - \sum_{\ell < j} \frac{Gm^2}{r_{\ell j}} \\
 & + \sum_{\ell < j} \frac{Gm^2}{2r_{\ell j}} [\vec{v}_\ell \cdot \vec{v}_j + (\vec{n}_{\ell j} \cdot \vec{v}_\ell)(\vec{n}_{\ell j} \cdot \vec{v}_j)] \\
 & - \sum_{\ell < j} \frac{3Gm^2}{2r_{\ell j}} |\vec{v}_\ell - \vec{v}_j|^2,
 \end{aligned} \tag{10}$$

where  $r_{\ell j} = |\vec{q}_\ell - \vec{q}_j|$  and  $\vec{n}_{\ell j} = (\vec{q}_\ell - \vec{q}_j)/r_{\ell j}$ . What we must do is replacing the potential by  $V$  and integrating over momenta after rewriting  $v$  by  $p$  in  $V$ . The technique for evaluating the integration is shown in [4]. The result is, however, simply interpreted as follows. If we approximate the system as an ideal gas of the particles, we can see that the averaged value for the particle velocity is given by

$$\overline{v^2} = \frac{3T}{m}. \tag{11}$$

Therefore the averaged potential is

$$\overline{V} = - \sum_{\ell < j} \frac{Gm^2}{r_{\ell j}} - \sum_{\ell < j} \frac{9GmT}{r_{\ell j}}. \tag{12}$$

Using the result of the last section, we get

$$Z_G = \int [D\xi] \exp \left\{ - \int d^3x \left[ \frac{1}{2} (\nabla\xi)^2 - z \left( \frac{m}{2\pi\beta} \right)^{3/2} \exp \left( -2m\sqrt{\pi G(\beta + 9/m)\xi} \right) \right] \right\}. \tag{13}$$

In the same manner, we can consider the case that particles have the electric charge  $q$  and the scalar charge  $\sigma$ . Then the averaged potential becomes

$$\overline{V} = - \sum_{\ell < j} \frac{Gm^2 - q^2 + \sigma^2}{r_{\ell j}} - \sum_{\ell < j} \frac{3T(3Gm^2 - \sigma^2)}{m r_{\ell j}}. \tag{14}$$

The partition function is now obtained as

$$\begin{aligned}
 Z_G = & \int [D\xi] \\
 & \exp \left\{ - \int d^3x \left[ \frac{1}{2} (\nabla\xi)^2 - z \left( \frac{m}{2\pi\beta} \right)^{3/2} e^{-2\sqrt{\pi[\beta(Gm^2 - q^2 + \sigma^2) + 3(3Gm^2 - \sigma^2)/m]\xi}} \right] \right\}.
 \end{aligned} \tag{15}$$

If the critical relation (BPS condition)  $Gm^2 - q^2 + \sigma^2 = 0$  is satisfied, the density is written as

$$\rho = z \frac{\partial}{\partial z} \ln Z_G = z \left( \frac{m}{2\pi\beta} \right)^{3/2} \left\langle \exp \left( -2\sqrt{\pi[3(3Gm^2 - \sigma^2)/m]\xi} \right) \right\rangle, \tag{16}$$

while the energy density is

$$\begin{aligned}\epsilon &= -\frac{\partial}{\partial\beta} \ln Z_G = \frac{3}{2}z \left(\frac{m}{2\pi\beta}\right)^{3/2} \frac{1}{\beta} \left\langle \exp\left(-2\sqrt{\pi[3(3Gm^2 - \sigma^2)/m]}\xi\right) \right\rangle \\ &= \frac{3}{2}\rho T.\end{aligned}\tag{17}$$

The relation between the density and the energy density is that of the ideal gas.

## 4 The equation for isothermal spheres

Our motivation is to study the properties of the system by the field theoretical method. In this talk, however, we concentrate ourselves on the equation for the dynamically equilibrium

configuration. Rescaling variables, we get the equation for an isothermal sphere:

$$\frac{1}{r^2} \frac{d}{dr} r^2 \frac{d}{dr} \psi = \mu^2 e^{-\psi}.\tag{18}$$

Here the characteristic scale (of mass)  $\mu$  is

$$\mu^2 = 4\pi z \left(\frac{1}{2\pi}\right)^{3/2} Tm \left[ (Gm^2 - q^2 + \sigma^2) \sqrt{\frac{m}{T}} + 3(3Gm^2 - \sigma^2) \sqrt{\frac{T}{m}} \right].\tag{19}$$

Without solving the equation, we can see that when  $Gm^2 - q^2 + \sigma^2 > 0$  and  $3Gm^2 - \sigma^2 < 0$ , the static configuration is impossible at sufficiently high temperature because  $\mu^2$  becomes negative at high temperature.

If we regard the particles as charged dilatonic black holes [5], this situation can occur when  $a^2 > 3$  (where  $a$  is the dilatonic coupling constant). This fact corresponds to the statement that black holes look like “elementary particles” if  $a^2 > 3$  [6].

## 5 Conclusion

In this talk, I have shown almost nothing to do with the property of the system governed by the long range forces, but the possible method to approach the study in terms of the effective field theory. For nearly BPS particles, the inclusion of the velocity-dependent interactions is essential. The field theoretic study of this exotic matter system is under investigation, which is very parallel to [1].

We note that since we deal with the grand canonical formalism, we can take the coalescence of particles (i.e. the change and transfer in the particle number) into consideration. This consideration is regarded as a model of the system of interacting black holes.

As we treat isothermal system, we wish to take the temperature as the Hawking temperature in the presence of a black hole. To incorporate such a background effect, we must extend the formulation to the case with fully curved spacetime and inclusion of radiation (and other possible fields). The work is in preparation.

## Note added

We have developed semi-general-relativistic formalism for the gravitating particles since after the workshop. We consider the Hamiltonian of Einstein-Hilbert gravity coupled to particles. Then we integrate out the momenta of particle using gaussian approximation. If the transfer energy between particles is negligible, the partition function can be represented by the path integral of the variables.

Here, for simplicity, the shift vector is assumed to vanish. Then the grand canonical partition function in  $(1 + d)$ -dimensional spacetime is written as:

$$Z_G = \int [DN] [D\gamma_{ij}] \exp \left\{ \int \left[ \frac{\beta}{16\pi G} N {}^{(d)}R + z \left( \frac{m}{2\pi\beta N} \right)^{d/2} e^{-\beta Nm} \right] \sqrt{\gamma} d^d x \right\}, \quad (20)$$

where  $N$  is the lapse function,  $\gamma_{ij}$  is the  $d$ -dimensional metric and  ${}^{(d)}R$  is the scalar curvature constructed from  $\gamma_{ij}$ .

The "classical" equation for  $N$  is

$$\nabla^2 N = 4\pi G \left( 2 \frac{d-2}{d-1} N m + \frac{d^2}{d-1} \frac{1}{\beta} \right) z \left( \frac{m}{2\pi\beta N} \right)^{d/2} e^{-\beta Nm}. \quad (21)$$

We emphasize that the  $O(v^2)$  velocity-dependent interaction is *automatically* incorporated.

The gravitational potential is involved in  $N$ . The coefficient of the  $1/\beta$  seemingly differs from the previous approach. But this is not the real defect because there is  $N^{d/2}$  in the right hand side and the gravitational potential is also included in the laplacian in the left hand side through  $\gamma_{ij}$ . The correct flat-space limit yields the agreement with the previous result.<sup>2</sup>(The interpretation is possible that the half of the averaged energy of a particle is shared with the gravitational potential, which cannot be neglected in this case.)

Taking other forces into account is straightforward.

One more comment is in order: even in the  $(1 + 2)$ -dimensions, isothermal spheres can exist! (but global topology of space seems to be curious.)

Further development and field theoretic study in this line will be reported elsewhere.

## References

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<sup>2</sup>To see this, substitute the metric (for  $d \neq 2$ ) as  $\gamma_{ij} \rightarrow N^{-\frac{2}{d-2}} \gamma_{ij}$  in eq. (20).

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