

# Wilson Loops on Torus

and

## Free Energy for Strings

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1. Introduction
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as Polyakov's path integral
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# 1. Introduction

Search for  
Unified Theory including Gravity  
in Higher dimension

Super String in 10 dim.

(bosonic in 26 dim)

Supergravity  $\rightarrow$  Kaluza-Klein Idea

Dimensional reduction<sup>①</sup>  
{ Compactification }  
to 4 dim.

Super Y-M Theory

A Large Gauge Sym.

must be BROKEN<sup>②</sup> down to

"realistic" symmetry

$U \times U \times U \times U \times U \dots$

① & ② may occur simultaneously.

"Wilson Loop Mechanism" may work.

On the other hand,

What extent can it have  
reference to **COSMOLOGY**?

which symmetry is realized  
&  
the symmetry will be restored  
in hot early Universe?

They are determined by  
examination of  
VACUUM & free energies.

## 2. Free energy of multi-particle gas

unlimitedly massive particle

in (Kaluza-Klein  
String Theories  
with level density  $\rho(m)$ )

Free energy for one particle (real scalar boson)  
mass =  $m$

$$F_b = \frac{V}{\beta} \int \frac{d^p k}{(2\pi)^p} \ln(1 - e^{-\beta \omega_k})$$

where temperature  $T = \beta^{-1}$

$$\omega_k = (k^2 + m^2)^{1/2}$$

$p$ : total dimensionality

This is also written, by use of

"Schwinger parameter", as

$$F_b = - \sum_{n=1}^{\infty} \int_0^{\infty} \frac{ds}{s} (2\pi s)^{-p/2} e^{-\frac{s}{2} m^2 - \frac{p^2 n^2}{2s}}$$

For multi-particle gas

$$F = \sum_i \int dm_i \rho(m_i) F_i(m_i)$$

$$\approx \sum_{i=1}^{\infty} d_i(m_i)$$

one-loop vacuum energy

$$= -\frac{1}{2} \int_0^{\infty} \frac{ds}{s} \int \frac{d^D k}{(2\pi)^D} e^{-\frac{s}{2}(k^2+m^2)}$$

$k^2 = |k^2 + k_0^2$

↓ Compactification of time direction on torus



$$\left\{ \begin{array}{l} \frac{1}{\beta} \sum_{n \rightarrow \infty} \int \frac{d^{D-1} k}{(2\pi)^{D-1}} \\ k_0 = \frac{2\pi n}{\beta} \end{array} \right.$$

Carry out

- Gaussian Integration on  $k$ 's
- transformation using the relation

$$\sum_{n \rightarrow \infty} e^{-n^2 \pi t} = \frac{1}{\sqrt{t}} \sum_{n \rightarrow \infty} e^{-\frac{\pi n^2}{t}}$$

Then

→ "vacuum energy" + "free energy"

$$\frac{1}{2} \int \frac{d^D k}{(2\pi)^D} \omega_k + \frac{1}{\beta} \int \frac{d^D k}{(2\pi)^D} \ln(1 - e^{-\beta \omega_k})$$

$$g(x,t) \equiv \sum_{n=-\infty}^{\infty} e^{-4\pi^2 n^2 t} \cos(2\pi n x)$$

$$= \frac{1}{\sqrt{4t}} \sum_{n=-\infty}^{\infty} e^{-(x-n)^2/4t}$$

$g$  is the solution of the diffusion equation

$$\frac{\partial}{\partial t} u = \frac{\partial^2}{\partial x^2} u$$

with an initial condition

$$u(t=0, x) = \sum_{n=-\infty}^{\infty} \delta(x-n)$$

Jacobi's theta functions with imaginary argument are also the solutions of some diffusion eq. with various initial conditions.

↓

Jacobi's imaginary transformation

### 3. Wilson loop mechanism

on <sup>non</sup> simply connected manifold

There can be non trivial Wilson loop elements

$$P \exp \int dx^I A_I \neq 1$$

SS

non zero vacuum expectation value

$$\langle A_I \rangle \neq 0$$

adjoint <sup>SS</sup> rep. Higgs field up to gauge transf.

#### ⊙ vacuum gauge field on torus

covariant derivative

$$D_I = \partial_I - i A_I$$

For stability SU(2) fundamental rep. scalar

$$-\Delta + m^2 \rightarrow k^2 + m^2$$

$$-\Delta - \partial_I^2 + m^2 \rightarrow k^2 + \frac{l^2}{r^2} + m^2$$

$$-\Delta - D_I^2 + m^2 \rightarrow k^2 + \frac{1}{r^2} (l \pm \phi)^2 + m^2$$

$$\phi \sim r A_I$$

$$A_I = A_I \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

I also considered

non simply connected manifolds

$$S^H/\mathbb{Z}_2$$

and evaluated the difference of

energy at zero and finite temperature.

## \* vacuum energy

$$\propto - \sum_{l=1}^{\infty} \frac{1}{l^3} \cos(2\pi\phi l)$$

This factor is just  
the trace of Wilson loop ~~about~~  
"  $\text{tr exp } \oint dx^2 A_2$  "  
and  $l$  is the "winding number"

## \* free energy

$$\propto -2 \sum_{l=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{\left(\frac{p^2}{(2\pi)^2} n^2 + l^2\right)^{D/2}} \cos(2\pi\phi l)$$

These energy are regarded as potential  
for  $\phi$  of which minimum determines  
the vacuum expectation value  $\langle\phi\rangle$ .

At zero & finite temperature, the contribution  
from the lowest winding Wilson-loop is dominant.

→ the pattern of symmetry is unchanged  
even if temperature increases.

4. Free energy for strings as

Polyakov's path integral (Polchinski)

the Time direction as torus

ansatz

$$X^0(\sigma_1, \sigma_2) = X^0(\sigma_1, \sigma_2) + \beta n \quad n \in \mathbb{Z}$$

$\Downarrow$

$$X^0 = \bar{X}^0 + (\text{oscillations})$$

$$\bar{X}^0 = \beta n \sigma_2 \quad [\text{zero-mode piece}]$$



Action  $S \rightarrow S_{\text{osci.}} + \bar{S}$

$$S = \frac{T}{2} \int d\sigma_1 d\sigma_2 \partial_a X^\mu \partial_b X^\nu$$

$$\text{with } d\sigma^2 = g_{ab} d\sigma^a d\sigma^b = |d\sigma_1 + \tau d\sigma_2|^2 \quad 0 \leq \sigma_{1,2} \leq 1$$

then,

$$\exp(-\mathcal{J}) = \exp\left(-\frac{\beta^2 h^2 T}{2\alpha'}\right)$$

This factor from zero mode

with sum over  $n$

reproduces the expression of free energy

TORUS Compactification of bosonic string



~~two~~

two  $U(1)$  charges

I showed two chemical potentials corresponding to their charges can be introduced by considering background geometry and antisymmetric tensor field with appropriate ansatz analogous to Kaluza-Klein theory.



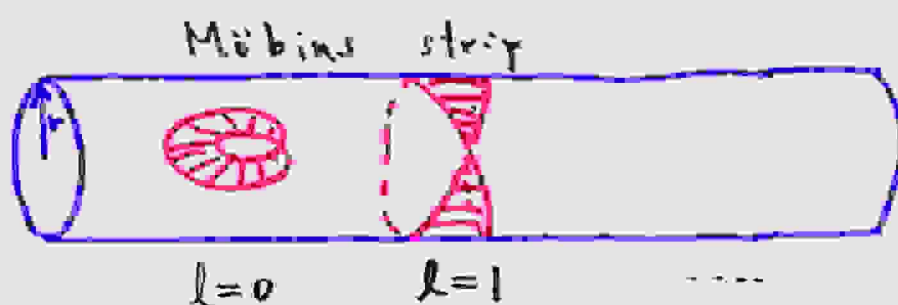
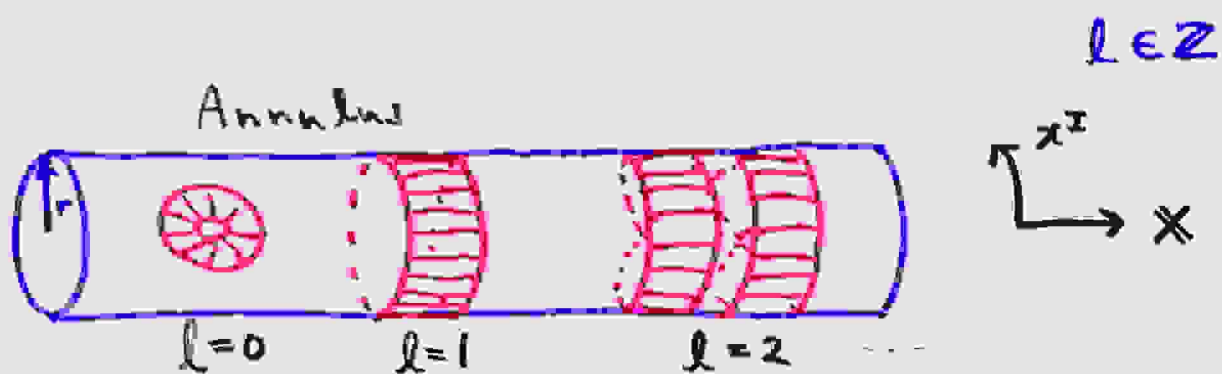
# 5. Open Strings and Wilson Loops.

## Open String on Torus

$$X^I(\sigma_1, \sigma_2) \quad 0 \leq \sigma_1, \sigma_2 \leq 1$$

on torus

$$X^I(\sigma_1, \sigma_2 + 1) = X^I(\sigma_1, \sigma_2) + 2\pi r l$$



Action

(Callan et al. ...)

$$G_{\mu\nu} = \begin{pmatrix} g & & \\ & -1 & \\ & & -1 \end{pmatrix}$$

$$S = \frac{T}{2} \int d\sigma \sqrt{-g} g^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X_\mu + iT \int ds A_\mu \frac{\partial X^\mu}{\partial s}$$

Back Ground Gauge Field  $A_I (\neq 0)$

\* boundaries



A : two



M : one

$$ds^2 = g_{ab} d\sigma^a d\sigma^b = d\sigma_1^2 + t^2 d\sigma_2^2$$

$$X^I = \bar{X}^I + (\text{oscillators})$$

$$\bar{X}^2 = 2\pi r l \sigma_2$$

$$S = \bar{S} + S_{\text{osc.}}$$

$$\exp(-\bar{S}) = \exp\left[-\frac{(2\pi r)^2 T}{2t} l^2\right] \underbrace{\Pi}_{\text{Boundary}} \text{tr} \exp\left(i \underbrace{L_B}_{\text{length of boundary}} 2\pi r T A_2 l\right)$$

$$\begin{cases} \text{Annulus } L_B = 1 \\ \text{Möbius } L_B = 2 \end{cases}$$

Consider gauge group  $SO(N)$  for simplicity

$$A_2 = A \left( \begin{array}{cc|ccc} 0 & -i & & & \\ i & 0 & & & \\ \hline & & & & \\ & & & & \\ & & & & 0 \end{array} \right) \Bigg\} N$$

$\underbrace{\hspace{10em}}_N$

$$\text{Annulus : } \Pi \text{tr exp}(\ ) = N^2 - 4N + 6 + 4(N-2) \cos(2\pi\phi l) + 2 \cos(4\pi\phi l)$$

$$\text{Möbius : } = N - 2 + 2 \cos(4\pi\phi l)$$

Sum over  $l$

( $\phi = rTA$ )

$$\sum_{l=-\infty}^{\infty} \exp\left[-\frac{(2\pi r)^2 T l^2}{2t}\right] \cos(2\pi\phi l)$$

$$= \sqrt{\frac{t}{2\pi Tr^2}} \sum_l \exp\left[-\pi t \frac{1}{2\pi Tr^2} (l-\phi)^2\right]$$

$t$  (after properly scaled) will be a "Schwinger parameter" in free energy written as path-integral form.

→ "Kaluza-Klein mode of excitation"  
mass<sup>2</sup>  $\propto (l-\phi)^2$

is added to

each "String oscillation" mode.

for even mode of oscillation

$$\frac{1}{2} (\oplus - \ominus) = \frac{(N-2)(N-3)}{2} + 1 + 2(N-2) \cos(2\pi\phi l)$$

$SO(N-2)$   
adj.

for odd mode

$$\frac{1}{2} (\oplus + \ominus) = \frac{(N-2)(N-1)}{2} + 1 + 2(N-2) \cos(2\pi\phi l) + 2 \cos(4\pi\phi l)$$

$SO(N-2)$   
symmetric  
rep.

∴  $\phi \neq 0 \pmod{2\pi}$ ,  $SO(N) \rightarrow SO(N-2)$

additional rotation of mass level

$SO(N)$  adj.

$$\partial_I A_i + [A_I, A_i]$$

$SO(N)$  sym. rep.

$$\partial_I \Phi + \{A_I, \Phi\}$$